Review for Midterm I[1]

Assigned: February 24, 2021

Multivariable Calculus MATH 53
with Professor Stankova

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1 Definitions

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, and to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **graph** examples for each definition. What is/are:

1. a *parametric* curve? paramteric equations? a *parameter*?
   How do they differ from their *Cartesian* counterparts?
2. a *cycloid*? Can it be parametrized? How?
3. an *ellipse*, a *hyperbola*, a *parabola*? How can each be parametrized?
4. *polar* coordinates? Relation to Cartesian coordinates? How do they differ?
5. a *polar* curve? Can *all* curves be represented as polar curves? In this context, what are the *Archimedian* spiral, the 4-*leaf rose*, the *figure*-8 *lemniscate*, a *cardioid*, an *asteroid*?
6. a *tangent line* to a parametric curve? to a polar curve? to a Cartesian curve?
   Do these tangents lines differ or are they the same line?
7. the *concavity* of a parametric curve?
8. the *arc length* of a parametric curve? of a polar curve?
9. a 3-*dimensional* coordinate system? What is $\mathbb{R}^2$? $\mathbb{R}^3$?
10. a vector $\vec{v}$ in $\mathbb{R}^2$ or in $\mathbb{R}^3$ from a geometric and from an algebraic point of view?
11. the *sum* and the *difference* of two vectors $\vec{v}$ and $\vec{w}$: geometrically and algebraically?
    How about the *scalar product* of a vector $\vec{v}$ with a scalar c?
12. the *basic properties* of addition, subtraction and scalar multiplication of vectors?
    How do these properties resemble properties of the corresponding operations on *real numbers*?
13. the *length* of a vector $\vec{v}$? How do we calculate it?
14. a *unit* vector? Can we re-scale all vectors to make them unit? How?

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15. the *standard* unit vectors in $\mathbb{R}^2$? in $\mathbb{R}^3$? Why are they so important?
16. a *linear combination* of vectors? How many ways can we express a vector as a linear combination of the standard unit vectors? Why?
17. a *median* and the *centroid* of a triangle? How are they related to our study of vectors?
18. the *dot product* of two vectors: algebraically and geometrically? How do we calculate it?
19. the *cross product* of two vectors: algebraically and geometrically? How do we calculate it?
20. the *basic properties* of dot and cross products, in relation to each other and in relation to the other operations on vectors?
21. *orthogonal* vectors? How to determine if two vectors are orthogonal, using vector operations?
22. *parallel* vectors? How do we determine if two vectors are pointing in the same direction, using operations on vectors? What are *colinear* vectors?
23. the *angle* between two vectors? How do we calculate the angle using the two vectors?
24. *orthogonal vector* and *scalar projections* of vectors? How do we calculate them?
25. *direction angles* and *direction cosines* of a vector?
    What is the main relationship between the three directional cosines of a vector?
26. a 2×2 matrix? a 3×3 matrix? the *determinants* of such matrices?
27. *co-planar* vectors? When are three vectors co-planar?
28. the *triple scalar product* of vectors? What is it useful for?
29. the *right-hand rule*? What is it used for?
30. the *point-slope* formula? What kind of a geometric object does it describe?
31. *parametric*, *vector*, and *Cartesian* (*symmetric*) equations for a line in space?
32. the *direction numbers of* a line in space? Are they unique? Why do we need 3 such numbers?
33. a *normal* vector to a plane? *vector*, *scalar*, and *linear* equations for a plane in space?
34. the *angle* between two planes? the *distance* between two planes?
35. a *cylinder*? What are its *base curve*, its *traces*, and its *ruling*?
    A cylinder can be thought of as the *disjoint union* of what objects? In how many ways?
36. a *quadratic equation* in three variables?
    How do we transform it into the standard equations of *quadric surfaces*?
37. the equations for *quadric cylindrical and non-cylindrical surfaces*? How many are they? How do we recognize each? What are their all possible *traces*? Why name them the way we do?
38. a *scalar* vs. a *vector* function? How do the latter relate to parametric curves?
39. the *limit*, *continuity*, *derivative*, and *integral* of a vector function?
    Why do we say that they are defined *component-wise*?
40. the *tangent* vs. a *secant* slope at a point P on the graph of a function $y = f(x)$?
    What is their relation to the derivative $f'(x)$ at P?
41. a *helix*? the *twisted cubic*? On which famous surfaces does each lie?
42. the *tangent vector* and *tangent line* for a vector function? the *unit* tangent vector?
43. the *arc-length* of a parametric curve? the arc-length *function*? How does it relate to velocity and speed? What does it mean to *re-parametrize* wrt arc-length? Why is this parametrization called "universal"? In what ways is it *not* unique?
44. an *intrinsic* feature of a curve? an *extrinsic* feature of a curve? a feature that is *independent* or not of parametrization? Can you list all features of curves we have studied and split them into intrinsic and extrinsic ones?
45. a *smooth* curve? What can we define on a smooth curve that cannot be well-defined on a non-smooth curve? Are the circle, any helix, and the twisted cubic smooth? How about any of the three *projections* of the twisted cubic onto the coordinate planes?
46. the *curvature* of a smooth curve? How does it depend on the parametrization of the curve? How does it relate to the derivative of any unit tangent vector? to the tangent vector wrt to

arc-length parametrization? to the derivative and acceleration vectors for any parametrization of the curve?

47. the curvature of a circle? of a helix? of a plane curve? What are the extreme curvatures along the twisted cubic or the regular cubic $y = x^3$?
48. the *normal* and *binormal* vectors of a vector function? How do they relate to the (unit) tangent vector? Which of them is independent of the parametrization?
49. the *normal*, *osculating*, and *rectifying* planes of a vector function? How do we picture them in relation to the motion of a particle along the corresponding path? Which famous vector is normal to each plane? What famous vectors does each plane contain?
50. the *tangential and normal components* of acceleration? On what does each component depend? Which component is necessarily non-negative and why? Which component can be negative and when does this happen? In which plane does the acceleration vector always lie? Is acceleration independent of parametrization?

2 Theorems

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be sure to understand, distinguish and **state** the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems. The latter means: decide which theorem to use, check (in writing!) that all conditions of your theorem are satisfied in the problem in question, and then state (in writing!) the conclusion of the theorem using the specifics of your problem.

1. **Conversions back-and-forth between Cartesian and polar coordinates.**
2. **Formulas for the following features of parametric curves and by polar curves:**
   - slopes of tangents to such curves;
   - second derivatives;
   - areas described by these curves, and areas between two such curves;
   - arc lengths of such curves;
   - surface area of solid of revolution given by such a curve.
3. **Properties of the following operations on vectors** (separately and in combinations):
   - addition, subtraction, scalar multiplication; taking linear combinations;
   - taking the magnitude of a vector;
   - dot product; cross product.
4. **Specials vector relations in a triangle**: formulas for the medians and for the sum of vectors from the centroid to each of the three vertices.
5. **Law of Cosines in a Triangle.**
6. **Formula for the dot product:** $\vec{v} \circ \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos \alpha$.
   - Corollary on how to calculate the angle between two vectors.
   - Conditions for vectors to be orthogonal, pointing in the same or opposite directions, making an acute or an obtuse angle.
7. **Formulas for vector and scalar orthogonal projections** of $\vec{v}$ onto $\vec{w}$.
8. **Formulas for direction angles and direction cosines**:
   - Expressing a vector using its direction cosines.
   - Sum of squares of the direction cosines.
9. **The Triangle Inequality:** when is equality obtained?
10. **Formulas for determinants of 2×2 and 3×3 matrices.**
    - Connection to cross products of vectors.
    - Basic properties of determinants that are relevant to cross products.

11. **Formula for the length of the cross product:** $|\vec{v} \times \vec{w}| = |\vec{v}| \cdot |\vec{w}| \sin\alpha$.
    - Corollary on how to calculate the angle between two vectors.
    - Conditions for vectors to be orthogonal, parallel, or making an acute or an obtuse angle.
12. **Formulas for:**
    - the areas of a parallelogram and a triangle; • for the volume of a parallelepiped;
13. **Condition for three vectors to be co-planar:** iff $\vec{v} \circ (\vec{w} \times \vec{u}) = 0$.
14. **Equations for lines:**
    - in the plane: point-slope formula; Cartesian equation;
    - in space: parametric, vector and Cartesian (symmetric) equations.
15. **Equations for planes:** vector, scalar, and linear equations.
    - the angle between planes;
    - distances from a point to a plane, and from a line to a plane, and between planes.
16. **Standard equations for quadric surfaces:**
    - quadric cylinders; • non-cylindrical quadric surfaces.
17. **Component-wise formulas for vector functions:**
    - limits, derivatives, integrals; • tangent and unit tangent vector.
18. **Trigonometric identities.**
    (a) Half-angle formulas (deg. reduction): $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.
    (b) Double-angle formulas: $\sin 2x = 2\sin x \cos x$, $\cos 2x = \cos^2 x - \sin^2 x$.
    (c) Turning products into sums: $\sin x \cos y = \frac{1}{2}(\sin(x - y) + \sin(x + y))$;
    $\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$; $\cos x \cos y = \frac{1}{2}(\cos(x - y) + \cos(x + y))$.
19. **Some formulas for arc-length and surface of revolution:**
    - If $f(x)$ is a function on $[a, b]$ such that $f'(x)$ is continuous on $[a, b]$, then the *arc length* of the curve $y = f(x)$ is $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$; and the *surface area* S of the solid obtained by revolving $y = f(x)$ about the x-axis is $S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$.
    - If $\vec{r}(t)$ is a vector function traced once by $t \in [a, b]$, then its arc length is $L = \int_a^b |\vec{r}\,'(t)| dt$.
20. **Differentiation Laws for vector functions:** DL$\pm$; PR$\cdot c$; PR$\cdot f(t)$; PR$\circ$; PR$\times$; CR.
21. **Unit tangent, normal, binormal vectors:** If $\vec{r}(t)$ is a *smooth* vector function, then
    - $\overrightarrow{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$; • $\overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{|\overrightarrow{T}'(t)|}$; • $\overrightarrow{B}(t) = \overrightarrow{T}(t) \times \overrightarrow{N}(t)$.
22. **Fundamental problems for vector functions:** (assuming all vectors below exist)
    - If $|\vec{r}(t)|$ is a constant, how are the tangent $\vec{r}\,'(t)$ and the position vector $\vec{r}(t)$ related?
    - How about the unit tangent vector $\overrightarrow{T}(t)$ and its derivative $\overrightarrow{T}'(t)$?
    - If $\vec{r}(t) \neq \vec{0}$ and $\vec{r}\,'(t)$ exists, what is $|\vec{r}(t)|'$?
    - Why is $|\overrightarrow{T}(t) \times \overrightarrow{T}'(t)| = |\overrightarrow{T}'(t)|$? Connection with $\overrightarrow{N}(t)$ and $\overrightarrow{B}(t)$?
23. **Relations with the arc-length function** $s(t)$**:**
    - $s'(t) = |\vec{r}\,'(t)|$ is the speed along the curve;
    - $\vec{r}\,'(t) = s'(t) \cdot \overrightarrow{T}(t)$; • $\vec{r}\,''(t) = s''(t) \cdot \overrightarrow{T}(t) + s'(t) \cdot \overrightarrow{T}'(t)$;
    - $\overrightarrow{T}(s) = \vec{r}\,'(s)$; • $L = \int_a^b |\overrightarrow{T}(s)|\, ds$; • $\kappa(s) = |\overrightarrow{T}'(s)| = |\vec{r}\,''(s)|$.
24. **Curvature:** The curvature of a vector function $\vec{r}(t)$ is given by $\kappa(t) = \frac{|\overrightarrow{T}'(t)|}{|\vec{r}\,'(t)|} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$.

    In the special case of a plane curve $y = f(x)$, the curvature equals $\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$.
25. **Coordinate planes "in motion":** $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ where (x_0, y_0, z_0) is in the plane, and $\langle a, b, c\rangle = \overrightarrow{T}, \overrightarrow{B}, \overrightarrow{N}$ for the normal, osculating, and rectifying planes.
26. **Acceleration:** $\vec{a}(t) = a_T \overrightarrow{T} + a_N \overrightarrow{N}$. Moreover, with speed $\nu = s'(t)$ and curvature κ:
    - $a_T = \nu' = \frac{\vec{r}\,'(t) \circ \vec{r}\,''(t)}{|\vec{r}\,'(t)|}$; • $a_N = \kappa\nu^2 = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|}$.

3 Problem Solving Techniques

1. **Convert between coordinate systems**
   - $(x, y) \mapsto (\sqrt{x^2 + y^2}, \arctan \frac{y}{x})$ and $(r, \theta) \mapsto (r \cos\theta, r \sin\theta)$.
2. **Find Tangent Slopes/Lines to a Parametric Curve given by $y = y(t)$ and $x = x(t)$:**
   - If $\frac{y'(t)}{x'(t)}$ is well-defined (i.e., both top and bottom quantities exist, are finite numbers, and $x'(t) \neq 0$), this is the tangent slope.
   - If $x'(t) = 0$ but $y'(t) \neq 0$, we have a vertical tangent line.
   - If $x'(t) = 0 = y'(t)$, apply LH to $\frac{y'(t)}{x'(t)}$ and repeat the process for the resulting quotient.
3. **Change lengths of vectors by dot products:**
   - $|\vec{v}|^2 \mapsto \vec{v} \circ \vec{v}$;
   - $|\vec{v}| \mapsto \sqrt{\vec{v} \circ \vec{v}}$.
4. **Constructing a Plane**
   1. Let P, Q, R be the three points given.
   2. Construct the vectors $\vec{PQ}$ and $\vec{PR}$.
   3. Find the normal vector $\vec{n} = \vec{PQ} \times \vec{PR}$.
   4. Let $\vec{n} = \langle a, b, c \rangle$ and $P = (x_0, y_0, z_0)$. Then the plane containing P, Q, R is given by

      $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

      It is common practice to move all the constants to one side to obtain the simplified equation $ax + by + cz = d$.
5. **Find Distances**
   - Point to Point: The distance from a point P to a point Q is found by taking the magnitude of the vector from P to Q i.e. $|\vec{PQ}|$.
   - Point to Line: Let P be a point and L be a line.
     1. Choose a point Q on L and construct the vector $\vec{PQ}$.
     2. Find the vector $\vec{v}$ that is parallel to the line.
     3. The distance is $\frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|}$.
   - Point to Plane: Let P be a point and let $\mathcal{P}$ be a plane. Suppose the equation for the plane is $ax + by + cz + d = 0$ and $P = (x_0, y_0, z_0)$. Then the distance is given by

     $$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

   - Line to Line: Let L_1 and L_2 be the two lines.
     1. Check if the lines intersect. If so, then distance is 0, otherwise move on to step 2.
     2. Let $\vec{v_1}$ be the vector parallel to L_1 and $\vec{v_2}$ be the vector parallel to L_2.
     3. Let $\vec{n} = \vec{v_1} \times \vec{v_2}$. This vector is the normal vector to the plane containing L_1 and the plane containing L_2 (so these planes are parallel).
     4. Pick a point $Q = (x_2, y_2, z_2)$ on L_2 and use the vector $\vec{n}$ to write the equation for the plane $\mathcal{P}_2$ containing L_2: $ax + by + cz + d = 0$.
     5. Pick a point $P = (x_1, y_1, z_1)$ on L_1. Now apply the algorithm for point to plane using P and $\mathcal{P}_2$.
   - Plane to Plane: Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the two planes.
     1. Choose a point P on $\mathcal{P}_1$.
     2. Apply the algorithm for point to plane using P and $\mathcal{P}_2$.
6. **Graph Quadric Surfaces**

1. Given a general equation, complete the square for the necessary variables i.e. the variables that have a squared term.
2. First consider the graph of the same surface that is centered at the origin and then shift it accordingly.
3. By setting variables to appropriate constants, describe the resulting traces.
4. Note any values of x, y, or z that results in special traces e.g. a point.

7. **Find Arc Length**
   1. Let $\vec{r}(t)$ be a vector valued function defined for $\alpha \leq t \leq \beta$. Differentiate $\vec{r}(t)$ to obtain $\vec{r}'(t)$.
   2. Evaluate $|\vec{r}'(t)|$.
   3. Then arc length is $L = \int_\alpha^\beta |\vec{r}'(t)| dt$.
8. **Calculating Curvature**
   1. Let $\vec{r}(t)$ be the given vector valued function.
   2. Find $\vec{r}'(t)$ and $\vec{r}''(t)$.
   3. Then curvature is given by
   $$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}.$$
9. **Finding Components of Acceleration**
   1. Let $\vec{r}(t)$ be the given vector valued function.
   2. Find $\vec{r}'(t)$ and $\vec{r}''(t)$.
   3. Then $\vec{a}(t) = \vec{r}''(t)$ can be decomposed as $\vec{a}(t) = a_T\vec{T} + a_N\vec{N}$ where $a_T = \frac{\vec{r}'(t) \circ \vec{r}''(t)}{|\vec{r}'(t)|}$ and $a_N = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|}$.

4 Problems for Review

The exam will be based on Homework, Lecture, Section and Quiz problems. Review **all** homework problems, and all your classnotes and discussion notes. Such a thorough review should be enough to do well on the exam. If you want to give yourself a mock-exam, select 4 representative problems from various HW assignments, give yourself 40 minutes, and then compare your solutions to the HW solutions. If you didn't manage to do some problems, analyze for yourself what went wrong, which areas, concepts and theorems you should study in more depth, and if you ran out of time, think about how to manage your time better during the upcoming exam.

4.1 Single-Variable Calculus in Other Coordinates

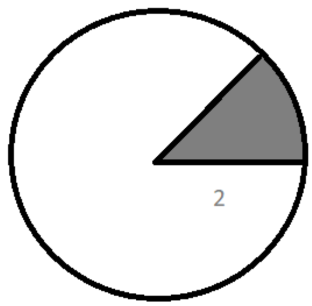
We looked at parametric representations of curves, polar coordinates, tangents to parametric and polar curves, and integral calculus (areas, arc lengths, surface areas of solids of revolution) on parametric and polar curves.

1. True/false practice:
   (a) The equations $x = r\cos\theta, y = 2r\sin\theta$ for some $r > 0$ represent an ellipse in polar coordinates.

   Solution: False. These equations are *parametric* equations for an ellipse; we're given x and y in terms of a parameter θ, and r is fixed here ("some $r > 0$"). Just because you see a θ doesn't immediately mean you're in polar coordinates.

   (b) The region $0 \leq \theta \leq \pi/4, 0 \leq r \leq 2$ is a circular sector.

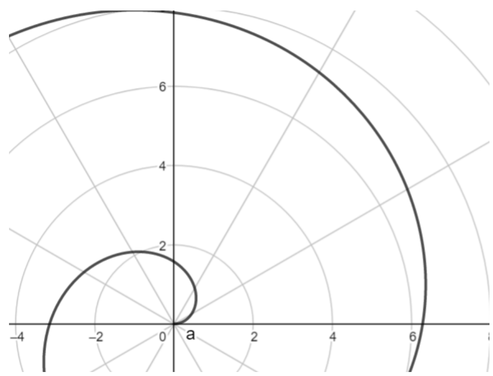
Solution: True. The region is $\frac{1}{8}$ of the circle with radius 2. You can think of regions like this as "pizza slices."



The circle $r = 2$ with the region $0 \leq \theta \leq \frac{\pi}{4}$, $0 \leq r \leq 2$ shaded in gray.

(c) In polar coordinates, instead of a vertical line test we have a "radial ray test;" a polar curve where you can connect the origin and two points on the curve with a straight line does not come from an expression of the form $r = f(\theta)$.

Solution: False. Consider the curve $r = \theta$. The points with polar coordinates $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{5\pi}{2}, \frac{5\pi}{2}\right)$ are both along the line $\theta = \frac{\pi}{2}$ and on the curve $r = \theta$, but our curve is still coming from an expression of the form $r = f(\theta)$.



The graph of $r = \theta$. Note that the radial ray from the origin in the direction $\theta = \frac{\pi}{2}$ intersects the curve twice.

(d) When finding the area between a parametric curve and the x-axis by setting up an integral, you should be careful with the bounds of your integral to ensure you get the right sign.

Solution: True. Consider as an example trying to find the area under the semicircle $x = \cos t$, $y = \sin t$, $0 \leq t \leq \pi$. We know since this semicircle lies above the x-axis that we want the area to be positive. But if we blindly set up our integral as

$$\int_0^\pi y(t)\frac{dx}{dt}dt = \int_0^\pi \sin t(-\sin t)dt = -\frac{\pi}{2},$$

we will get a negative answer. This is because our technique of finding the area under the cruve by finding the integral $[dt]$ of $y(t)\frac{dx}{dt}$ is coming from expressing y implicitly as a function of x and then using integration by substitution (i.e. making a change of variables) to express y as a function of t instead of as a function of x. In the cartesian setup, we want to be doing $\int_{-1}^1 y(x)dx$, and $x = -1$ when $t = \pi$, $x = 1$ when $t = 0$, so the correct parametric integral to write down is

$$\int_\pi^0 y(t)\frac{dx}{dt}dt = \int_\pi^0 -\sin^2 t dt = \int_0^\pi \sin^2 t dt = \frac{\pi}{2}.$$

In practice, if you've sketched the region you're trying to find the area of and know that it lies above the x-axis (so the answer should be positive) but the answer you get when you evaluate your parametric integral is negative, it means you've picked the wrong bounds on t.

(e) There is nothing mysterious at all about our formulas for the arc length of parametric and certain polar curves; they come from the usual cartesian formulas, our formulas for tangent slopes, and integration by substitution.

Solution: True. For example, we have for a function $y = F(x)$ the formula for arc length between $x = a$ and $x = b$ is

$$\int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

If this same curve is represented as $x = f(t), y = g(t)$, with $g(\alpha) = a, g(\beta) = b$, we can write this integral using integration by substitution as

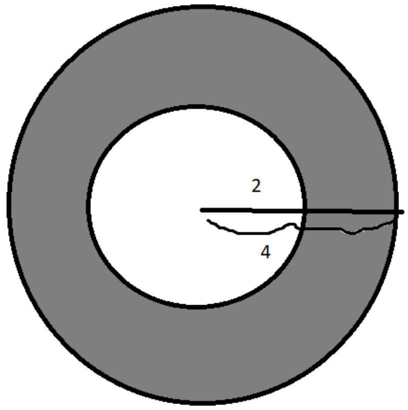
$$\int_{t=\alpha}^{t=\beta} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\frac{dx}{dt}dt = \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\frac{dx}{dt}dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

which is our formula for the arc length of a parametric curve.

Any parametric curve can be broken up as a number of pieces which we can write [implicitly] as $y = F(x)$ along each piece, so we can do the above analysis on a bunch of little pieces of our curve to make this analysis work even for parametric curves that don't pass the vertical line test.

The derivation for polar curves is similar and is an instructive exercise.

2. Sketch and describe in the words the regions in the plane defined by the following inequalities:
   (a) $2 \leq r \leq 4$;



Solution:
The region is the annulus (region between two circles) with inner radius 2 and outer radius 4.

(b) $\pi \le \theta \le 2\pi$.

Solution: The region is the entire xy-plane.

3. Find three distinct representations, one of which has $r < 0$, of the point with Cartesian coordinates $(1, \sqrt{3})$ in polar coordinates.

Solution: The cartesian coordinates $(1, \sqrt{3})$ have $x^2+y^2 = 1^2+(\sqrt{3})^2 = 4$, so they correspond to a radius of $r = 2$. We also have that $\arctan \frac{y}{x} = \arctan \frac{\sqrt{3}}{1} = \arctan \sqrt{3} = \frac{\pi}{3}$, so one polar representation of this point is $\boxed{(2, \dfrac{\pi}{3})}$. To find two more polar representations, we note that (r, θ) is the same point as $(-r, \theta+\pi)$, so $\boxed{(-2, \dfrac{4\pi}{3})}$ is another polar representation of this point. A third we can find by noting that (r, θ) is the same point as $(r, \theta + 2\pi)$, so $\boxed{(2, \dfrac{7\pi}{3})}$ is a third polar representation of this point.

4. What is the slope of the tangent line to the curve $r = 4\theta^2$ at the point with the Cartesian coordinates $(0, \pi)$?

Solution: The cartesian coordinates $(0, \pi)$ correspond to polar coordinates of $(\pi^2, \frac{\pi}{2})$. Our curve is given by an equation of the form $r = h(\theta)$, so we can apply our formula for the slope of a tangent line to a polar curve where r is a function of θ. We have

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\ &= \frac{8\theta\sin\theta + 4\theta^2\cos\theta}{8\theta\cos\theta - 4\theta^2\sin\theta} \\ &= \frac{4\pi\cdot 1 + \pi^2\cdot 0}{4\pi\cdot 0 - \pi^2\cdot 1} \\ &= \boxed{-\frac{4}{\pi}}.\end{aligned}$$

5. Find all points (x, y) in the plane where the curve $x = 2t^3 - 3t^2, y = t^4 - 4$ has horizontal or vertical tangents.

Solution: We know that the slope of the tangent line to the curve will be

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

For this curve, this tells us that

$$\frac{dy}{dx} = \frac{4t^3}{6t^2 - 6t}.$$

We will have a horizontal tangent if the slope is 0 and a vertical tangent if this expression for $\frac{dy}{dx}$ is of the form $\frac{a}{0}$, $a \neq 0$. (In the case of $\frac{0}{0}$, we need to try something else, perhaps L'hospital's rule, to figure out what is happening). We see that the numerator $\frac{dy}{dt}$ is 0 only when $t = 0$; at this time, the denominator is also 0, but we can cancel a factor of t from the numerator and the denominator to get $\frac{4t^2}{6t-6} = \frac{4 \cdot 0}{6 \cdot 0 - 6} = 0$. This tell us that we have a horizontal tangent when $t = 0$, at which point we are at the point $\boxed{(0, -4)}$ in the plane. The denominator $\frac{dx}{dt}$ is 0 when $t = 0$ or when $t = 1$. When $t = 0$, we have already seen that we have a horizontal tangent; when $t = 1$, we have a denominator of 0 and a nonzero numerator, so we have a vertical tangent when $t = 1$. This tells us that we have a vertical tangent at the point $\boxed{(-1, -3)}$ in the plane.

6. Use the formula for surface area of solids of revolution of parametric curves to prove that the surface area of a sphere of radius 1 is 4π.

Solution: The sphere of radius 1 is the solid of revolution formed by rotating the semicircle with radius 1 above the x-axis around the x-axis. We can parametrize this semicircle as $x = \cos t, y = \sin t, 0 \leq t \leq \pi$. Our formula for surface area of a solid of revolution is

$$\int_{\alpha}^{\beta} 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Using this, we see that the surface area of the sphere is

$$\begin{aligned} \int_0^{\pi} 2\pi \sin t \sqrt{\sin^2 t + \cos^2 t} dt &= 2\pi \int_0^{\pi} \sin t dt \\ &= -2\pi \cos t \Big|_0^{\pi} \\ &= -2\pi(-1) - (-2\pi \cdot 1) \\ &= 4\pi, \end{aligned}$$

so the surface area of a sphere of radius 1 is 4π.

7. Consider the curve C described in polar coordinates by $r = 2\sin\theta$.
   a) Sketch C in the xy-plane. Indicate the interval for θ over which the curve is traversed exactly once. Include all of your calculations.

   Solution: This is the circle centered at $(0, 1)$ with radius 1. The interval is $[0, \pi)$

   b) Find the area enclosed by C.

Solution: The area of the circle mentioned above is π. We can also see this by applying the formula

$$A = \frac{1}{2}\int_0^\pi r^2 d\theta = \int_0^\pi 2\sin^2\theta d\theta = \int_0^\pi 1 - \cos(2\theta) d\theta = \pi.$$

c) Prove that C is a circle by finding its Cartesian equation.

Solution: We have that

$$r = 2\sin\theta \Rightarrow r = 2y/r \Rightarrow r^2 = 2y \Rightarrow x^2 + y^2 - 2y = 0 \Rightarrow x^2 + (y-1)^2 = 1,$$

as desired

8. Derive the formula for the surface area of a sphere or radius $r = 5$ by following the steps below. Include all calculations and relevant explanations.

a) Sketch and parameterize a circle of radius $r = 5$ centered at the origin in the xy-plane.

Solution: The parametrization is $x(t) = 5\cos t$ and $y(t) = 5\sin t$ for $t \in [0, 2\pi)$.

b) Now revolve the circle about the y-axis, and using your parameterization from part (a), find the surface area of the resulting solid.

Solution:

$$S = \int_0^{2\pi} 2\pi y(t)\sqrt{(x'(t))^2 + (y'(t))^2} dt = 2\pi\int_0^{2\pi} 5\sin t\sqrt{25} dt = 50\pi\int_0^{2\pi} \sin t dt = 100\pi.$$

4.2 Vectors and the Geometry of Space

We looked at the geometry of three-dimensional space $\mathbb{R}^3$, vectors and vector addition, the dot product, scalar and vector projection, direction cosines, the cross product, the determinant formula for cross products, the scalar triple product, equations for lines in $\mathbb{R}^3$ (vector, parametric, symmetric), equations for planes in $\mathbb{R}^3$, intersections of planes with lines and planes with planes, distances from points to lines and planes.

1. True/false practice:

a) If you draw the positive x-axis going to the left along a sheet of paper and the positive y-axis going down towards the bottom of the sheet of paper, then to get a set of coordinate axes for $\mathbb{R}^3$, the positive z-axis would be coming straight up out of the paper.

Solution: True. This is what we get from the right-hand rule (by convention, we use only right-handed coordinate systems where the positive x, y, and z axes satisfy the right hand rule)

b) The expression $((\vec{a} \circ \vec{b})\vec{c}) \circ \vec{d}$ makes sense.

Solution: True. The expression $\vec{a} \bullet \vec{b}$ is a dot product of two vectors, which is allowed (and gives us a scalar as its output). $(\vec{a} \bullet \vec{b})\vec{c}$ is the scalar multiplication of the vector $\vec{c}$ by the scalar $\vec{a} \bullet \vec{b}$, which is allowed (and gives us a vector as the output). The overall expression, then, is the dot product of two vectors, which is allowed, so the expression makes sense.

c) The cross product is associative.

Solution: False. A counterexample: let $\vec{i}, \vec{j}, \vec{k}$ be the three standard unit vectors in the positive x, y, and z directions. $\vec{j} \times (\vec{j} \times \vec{k}) = \vec{j} \times \vec{i} = -\vec{k}$, while $(\vec{j} \times \vec{j}) \times \vec{k} = \vec{0} \times \vec{k} = \vec{0}$.

d) It makes sense to talk about *the* normal direction to a line in $\mathbb{R}^3$.

Solution: False. A line in $\mathbb{R}^3$ is described by a single vector in the direction of the line, as well as a point on the line. But we can have two non-parallel vectors that are both perpendicular to a line in $\mathbb{R}^3$; for example, the x axis is a line, and the vectors $\vec{j}, \vec{k}$ in the positive y and z directions are both perpendicular to it but are not at all the same direction.

e) One way to think about the equation $ax + by + c = 0$ as a line in the plane is that it comes form knowing the vector $\langle a, b\rangle$ is orthogonal to the line and using some point (x_0, y_0) on the line.

Solution: True. While this isn't the usual way of thinking about equations of lines in the plane, it is the 2-dimensional analogue of the expression $ax + by + cz + d = 0$ for a plane in $\mathbb{R}^3$. In particular, if we know that the vector $\langle a, b\rangle$ is perpendicular to a line containing (x_0, y_0), then we have for any point (x, y) on the line,

$$\langle a, b\rangle \bullet \langle x - x_0, y - y_0\rangle = 0.$$

Expanding this dot product, we get

$$ax - ax_0 + by - by_0 = 0,$$

or, equivalently,

$$ax + by + (-ax_0 - by_0) = 0,$$

giving us the form $ax + by + c = 0$ for our line as desired.

2. Let $\vec{u} = \langle 1, -1, 3\rangle, \vec{v} = \langle 0, 2, -1\rangle, \vec{w} = \langle 2, 0, 1\rangle$. Evaluate $\vec{u} \circ (\vec{v} \times \vec{w})$.

Solution: We use the fact that the cross product distributes over addition, so that

$$\begin{aligned}\vec{v} \times \vec{w} &= (2\vec{j} - \vec{k}) \times (2\vec{i} + \vec{k}) \\ &= 2\vec{j} \times (2\vec{i} + \vec{k}) - \vec{k} \times (2\vec{i} + \vec{k}) \\ &= 4\vec{j} \times \vec{i} + 2\vec{j} \times \vec{k} - 2\vec{k} \times \vec{i} - \vec{k} \times \vec{k}.\end{aligned}$$

We also know that

$$\vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j},$$

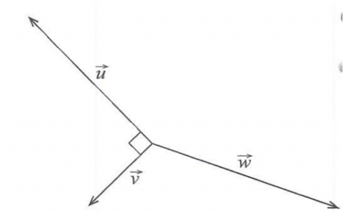
that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ for any vectors $\vec{a}$ and $\vec{b}$, and that the cross product of any vector and itself is $\vec{0}$. This gives

$$\begin{aligned}\vec{v} \times \vec{w} &= -4\vec{k} + 2\vec{i} - 2\vec{j} - \vec{0} \\ &= 2\vec{i} - 2\vec{j} - 4\vec{k}.\end{aligned}$$

We can now evaluate the dot product of $\vec{u}$ and this vector, using the fact that the dot product distributes over addition and that the dot product of any two elements of $\vec{i}, \vec{j}, \vec{k}$ is 0 if they are distinct and 1 if they are the same. This gives

$$\begin{aligned}\vec{u} \bullet (\vec{v} \times \vec{w}) &= (\vec{i} - \vec{j} + 3\vec{k}) \bullet (2\vec{i} - 2\vec{j} - 4\vec{k}) \\ &= 2\vec{i} \bullet \vec{i} - 2\vec{i} \bullet \vec{j} - 4\vec{i} \bullet \vec{k} - 2\vec{j} \bullet \vec{i} + 2\vec{j} \bullet \vec{j} + 4\vec{j} \bullet \vec{k} + 6\vec{k} \bullet \vec{i} - 6\vec{k} \bullet \vec{j} - 12\vec{k} \bullet \vec{k} \\ &= 2 - 0 - 0 - 0 + 2 + 0 + 0 + 0 - 12 \\ &= \boxed{-12}.\end{aligned}$$

3. Let the vectors in the figure satisfy $|\vec{u}| = 2$, $|\vec{v}| = 1$, and $\vec{u} + \vec{v} + \vec{w} = \vec{0}$.
   a) What is $|\vec{w}|$ equal to? Explain. (*Hint:* Express $\vec{w}$ in terms of the other vectors, and use this in other parts of the problem too!)



Solution: We see that $\vec{w} = -\vec{u} - \vec{v}$ so $|\vec{w}|^2 = (-\vec{u} - \vec{v}) \cdot (-\vec{u} - \vec{v}) = |\vec{u}|^2 + |\vec{v}|^2$ because $u \cdot v = 0$. Hence $|w| = \sqrt{1+4} = \sqrt{5}$.

   b) What is the dot product $\vec{v} \circ \vec{w}$ equal to? Explain.

Solution: $\vec{v} \cdot \vec{w} = \vec{v} \cdot (-\vec{u} - \vec{v}) = -|\vec{v}|^2 = -1$.

   c) What is the length of $\vec{v} \times \vec{w}$? Explain.
   (*Hint:* If α is the angle between the vectors, can you find $\cos\alpha$ from something above?)

Solution: From above $-1 = \vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\alpha = \sqrt{5}\cos\alpha$. Thus $\sin\alpha = \sqrt{1 - \cos^2\alpha} = \sqrt{1 - 1/5} = 2/\sqrt{5}$. We took the positive square root because $\alpha \in [0, \pi]$. Then $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\alpha = 2$.

4. Consider the line ℓ with symmetric equations given by

$$\frac{x-1}{2} = \frac{y+2}{3} = z.$$

   a) Is it parallel, perpendicular, or skew to the line whose parametric equations are $x = t + 4, y = 2 - t, z = t - 1$?

Solution: The first thing to do is to find the vector representing the direction of the line ℓ. The line ℓ is in the direction $\langle 2,3,1\rangle$, as we can see by noting that the signs on x, y, z in the symmetric equations above are all positive and that the denominators are 2, 3, and 1, respectively.
The second line is in the direction $\langle 1,-1,1\rangle$, as we can see by converting our parametric equations to the vector equation

$$\mathbf{r}(t) = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

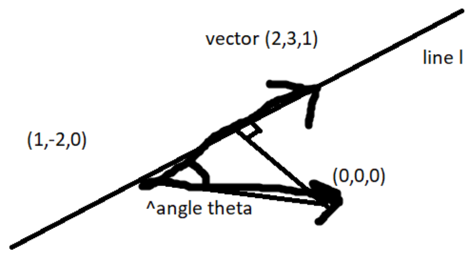
The two directions are not parallel, as they are not scalar multiples of each other. We see that $\langle 2,3,1\rangle \bullet \langle 1,-1,1\rangle = 2-3+1 = 0$; this does *not*, however, tell us that the two lines are perpendicular. Since we are working in $\mathbb{R}^3$, it is possible for two lines not to intersect at all without the lines being parallel (this is what it means to be skew). So we see if the two lines intersect.
Parametric equations for ℓ are $x = 1+2s, y = -2+3s, z = s$. To find a point of intersection of ℓ and this second line, we try to solve the system of equations

$$\begin{aligned} 1+2s &= t+4 \\ -2+3s &= -t+2 \\ s &= t-1. \end{aligned}$$

Solving the second two equations together, we see that $-2+3(t-1) = 3t-5$ must be equal to $-t+2$, so that $4t = 7$. Plugging $t = \frac{7}{4}$, $s = \frac{3}{4}$ into the first equation, we get $1 + 2 \cdot \frac{3}{4} = \frac{5}{2}$ on the left-hand side and $\frac{7}{4} + 4 = \frac{23}{4}$ on the right-hand side. This is inconsistent, so we see that this system of equations does not have a solution and thus that the two lines do not intersect.
Since the lines are nonintersecting and nonparallel, they are $\boxed{\text{skew}}$.

b) What is the distance from the origin to the line ℓ?

Solution: We will need to do a bit of trigonometry to figure this out, as we are trying to find the distance of the perpendicular segment from $(0,0,0)$ to the line ℓ. The situation is pictured below:



The setup for finding the distance from a point to a line.

To find the perpendicular distance, it is enough to find $\sin\theta$, where θ is the angle between the line ℓ, represented by the vector $\langle 2,3,1\rangle$ and the vector $\vec{v} = \langle -1,2,0\rangle$ from a point on the line to the origin. Knowing the sine of this angle and the length of the hypotenuse (in this case, $|\langle -1,2,0\rangle|$) gives us the length of the perpendicular segment from the origin to the line. But we can find $\sin\theta$ by using cross products! In particular, we have

$$\begin{aligned}
|\langle -1,2,0\rangle \times \langle 2,3,1\rangle| &= |\langle -1,2,0\rangle||\langle 2,3,1\rangle|\sin\theta \\
\left|\det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix}\right| &= \sqrt{1^2+2^2+0^2}\sqrt{2^2+3^2+1^2}\sin\theta \\
|2\vec{i} - (-1)\vec{j} - 7\vec{k}| &= \sqrt{5}\sqrt{14}\sin\theta \\
\sqrt{2^2+1^2+7^2} &= \sqrt{5}\sqrt{14}\sin\theta \\
\sqrt{54} &= \sqrt{5}\sqrt{14}\sin\theta.
\end{aligned}$$

We are looking for $|\langle -1,2,0\rangle|\sin\theta$, so we want $\sqrt{5}\sin\theta$, which is equal to $\frac{\sqrt{54}}{\sqrt{14}} = \frac{\sqrt{27}}{\sqrt{7}} = \boxed{\frac{3\sqrt{21}}{7}}$.

c) Write an equation for a plane perpendicular to this line.

Solution: We know the direction the line ℓ is traveling, and we know a point on ℓ. Since $(1,-2,0)$ is on ℓ, we can construct the plane containing $(-1,2,0)$ and normal to the vector $\langle 2,3,1\rangle$ to get a plane perpendicular to ℓ. The equation for this plane is

$$\begin{aligned}
\langle 2,3,1\rangle \bullet \langle x-1, y+2, z\rangle &= 0 \\
2x - 2 + 3y + 6 + z &= 0,
\end{aligned}$$

so that

$$\boxed{2x + 3y + z + 4 = 0}$$

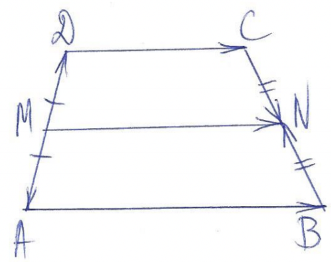
is the equation for our plane.

5. In a trapezoid $ABCD$ it is known that base AB is 10 cm and the base DC is 6 cm. Let M

be the midpoint of side AD and N the midpoint of side BC. The segment MN is called the *midsegment* of the trapezoid $ABCD$.

a) Write $\overrightarrow{MN}$ as a linear combination of $\overrightarrow{AB}$ and $\overrightarrow{DC}$ and, using vectors, prove that this linear combination is correct.

Solution:



Notice $\overrightarrow{MN}=\overrightarrow{MA}+\overrightarrow{AB}+\overrightarrow{BN}$ and $\overrightarrow{MN}=\overrightarrow{MD}+\overrightarrow{DC}+\overrightarrow{CN}$. Then

$$2\,\overrightarrow{MN}=\overrightarrow{MA}+\overrightarrow{MD}+\overrightarrow{AB}+\overrightarrow{DC}+\overrightarrow{BN}+\overrightarrow{CN}=\overrightarrow{AB}+\overrightarrow{DC}$$

so $\overrightarrow{MN}=\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{DC})$

b) Prove that the midsegment MN is parallel to the bases of the trapezoid. Find its length. (Hint: You may use part (a).)

Solution: Since $\overrightarrow{AB}\ //\ \overrightarrow{DC}$ and $|\ \overrightarrow{AB}\ |=10$ and $\overrightarrow{DC}=6$ so $\overrightarrow{AB}=\frac{5}{3}\ \overrightarrow{DC}$. Then $\overrightarrow{MN}=\frac{1}{2}(\frac{5}{3}\ \overrightarrow{DC}+\overrightarrow{DC})=\frac{4}{3}\ \overrightarrow{DC}$. Thus $\overrightarrow{MN}\ //\ \overrightarrow{DC}\ //\ \overrightarrow{AB}$ and $|\ \overrightarrow{MN}\ |=\frac{4}{3}\cdot 8=6$ cm.

6. Consider the three points $P=(1,-2,0), Q=(-1,0,3)$, and $R=(-3,2,0)$ in $\mathbb{R}^3$.

a) What is $\cos\angle PQR$?

Solution: We know that

$$\vec{QP} \bullet \vec{QR} = |\vec{QP}||\vec{QR}| \cos \angle PQR.$$

We compute the vectors $\vec{QP}$ and $\vec{QR}$ first by taking the vectors from Q to P and Q to R. We have

$$\begin{aligned} \vec{QP} &= \vec{P} - \vec{Q} \\ &= \langle 1, -2, 0 \rangle - \langle -1, 0, 3 \rangle \\ &= \langle 2, -2, -3 \rangle \\ \vec{QR} &= \mathbf{r} - \vec{Q} \\ &= \langle -3, 2, 0 \rangle - \langle -1, 0, 3 \rangle \\ &= \langle -2, 2, -3 \rangle. \end{aligned}$$

Each of these vectors has length $\sqrt{2^2 + 2^2 + 3^2} = \sqrt{4 + 4 + 9} = \sqrt{17}$. We can compute the dot product of the two vectors as

$$\begin{aligned} \vec{QP} \bullet \vec{QR} &= \langle 2, -2, -3 \rangle \bullet \langle -2, 2, -3 \rangle \\ &= 2(-2) + (-2)2 + (-3)(-3) \\ &= -4 - 4 + 9 \\ &= 1. \end{aligned}$$

This tells us that

$$\cos \angle PQR = \frac{\vec{QP} \bullet \vec{QR}}{|\vec{QP}||\vec{QR}|} = \frac{1}{\sqrt{17}\sqrt{17}} = \boxed{\frac{1}{17}}.$$

b) Find an equation for the plane containing the three points P, Q, and R.

Solution: To find a plane, we need a point and a normal vector. Since we know the three points P, Q, R in the plane, we know that the vectors $\vec{QP}$ and $\vec{QR}$ that we computed above lie in the plane. Given two vectors that are not parallel, we can cook up a third vector perpendicular to both of them by taking the cross product. Hence $\vec{QP} \times \vec{QR}$ will be perpendicular to the plane containing P, Q, and R. We compute the cross product

$$\begin{aligned} \vec{QP} \times \vec{QR} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -3 \\ -2 & 2 & -3 \end{vmatrix} \\ &= 12\vec{i} - (-12)\vec{j} + 0\vec{k} \\ &= 12\vec{i} + 12\vec{j}. \end{aligned}$$

So we want to find the plane perpendicular to the vector $12\vec{i} + 12\vec{j}$ containing the point $Q = (-1, 0, 3)$. Using the fact that $\langle 12, 12, 0\rangle$ and $\langle x+1, y, z-3\rangle$ must be perpendicular for any point (x, y, z) in the plane, we have

$$\begin{aligned} \langle 12, 12, 0\rangle \cdot \langle x+1, y, z-3\rangle &= 0 \\ 12x + 12 + 12y &= 0, \end{aligned}$$

so that the equation

$$\boxed{12x + 12y + 12 = 0}$$

is an equation for the plane containing P, Q, and R.
We can check our work by checking that P and R actually satisfy this equation; $12 \cdot 1 + 12 \cdot (-2) + 12 = 0$ and $12 \cdot (-3) + 12 \cdot 2 + 12 = 0$, so we are good.

c) What is the area of the triangle PQR?

Solution: We know that the magnitude of the cross product of two vectors can be interpreted geometrically as the area of the parallelogram with distinct sides given by the two vectors. In particular, the area of a triangle with two sides given by two vectors will be half the area of the corresponding parallelogram, so the area of the triangle PQR is

$$\frac{1}{2}|\vec{QP} \times \vec{QR}| = \frac{1}{2}|12\vec{i} + 12\vec{j}| = \frac{1}{2}\sqrt{12^2 + 12^2} = \frac{1}{2}\sqrt{288} = \frac{1}{2}(12\sqrt{2}) = \boxed{6\sqrt{2}}.$$

7. Consider the two planes $x + 2y + 2z + 4 = 0$ and $3x - 4y + 12z = 0$.

a) What is the cosine of the (acute) angle between the planes?

Solution: We find the cosine of the acute angle between the planes by finding the cosine of the angle between their normal vectors. The two planes' normal vectors are $\langle 1, 2, 2\rangle$ and $\langle 3, -4, 12\rangle$. The cosine of the angle is

$$\frac{\langle 1, 2, 2\rangle \bullet \langle 3, -4, 12\rangle}{|\langle 1, 2, 2\rangle||\langle 3, -4, 12\rangle|} = \frac{1 \cdot 3 + 2 \cdot (-4) + 2 \cdot 12}{\sqrt{1^2 + 2^2 + 2^2}\sqrt{3^2 + 4^2 + 12^2}} = \frac{3 - 8 + 24}{\sqrt{9}\sqrt{169}} = \frac{19}{39}.$$

b) Write symmetric equations of the line of intersection between the two planes.

Solution: To describe a line in $\mathbb{R}^3$, we need two pieces of data: a point on the line and a vector in the direction of the line. Let's first find a vector in the direction of the line. We know that the line, since it lies in both planes, is perpendicular to the two planes' normal vectors. So to find a vector in the direction of the line, it suffices to find a vector perpendicular to both $\langle 1,2,2\rangle$ and $\langle 3,-4,12\rangle$. We can find such a vector by taking a cross product:

$$\langle 1,2,2\rangle \times \langle 3,-4,12\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ 3 & -4 & 12 \end{vmatrix} = 32\vec{i} - 6\vec{j} - 10\vec{k},$$

so $\langle 32,-6,-10\rangle$ is parallel to the direction of our line.
We now find one point on the line of intersection. An easy place to look for a single point on a line is the line's intersection with one of the coordinate planes. We will try to find the point with $z = 0$ that lies on this line (in general, we know any line will intersect at least one of the three coordinate planes). So we want a point with z-coordinate zero lying on both the planes $x + 2y + 2z + 4 = 0$ and $3x - 4y + 12z = 0$. This means we need to solve the system of equations

$$\begin{aligned} x + 2y &= -4 \\ 3x - 4y &= 0. \end{aligned}$$

Solving this system of equations, we see that $x = \frac{-8}{5}$, $y = \frac{-16}{5}$ is the unique solution, so the point $(-\frac{8}{5}, -\frac{16}{5}, 0)$ lies on both planes and hence on the line of intersection of the two planes.
We now write symmetric equations for the line of intersection of the two planes, noting that this line goes through $(-\frac{8}{5}, -\frac{16}{5}, 0)$ and is in the direction $\langle 32,-6,-10\rangle$, by writing the parametric equations

$$\begin{aligned} x &= -\frac{8}{5} + 32t \\ y &= -\frac{16}{5} - 6t \\ z &= -10t \end{aligned}$$

and solving for t to get symmetric equations of the form

$$\boxed{\frac{x + \frac{8}{5}}{32} = \frac{-\frac{16}{5} - y}{6} = \frac{-z}{10}}.$$

8. Write an equation for the surface of revolution formed by rotating the curve $x = \sqrt{1 + y^2}$ about the x-axis. What kind of surface is this?

Solution: Rotating the curve about the x-axis through $\mathbb{R}^3$ means the points on the curve trace out through constant x planes circles of the form $x = \sqrt{1 + y^2 + z^2}$, as rotating a graph $x = f(y)$ about the x-axis replaces y with $\sqrt{y^2 + z^2}$. This is half of a hyperboloid of two sheets, since we can rearrange the equation to $x^2 - y^2 - z^2 = 1$, but we are only working with the piece $x = \sqrt{1 + y^2 + z^2}$ and not the piece $x = -\sqrt{1 + y^2 + z^2}$.

9. One line L_1 passes through the point $P(1,1,0)$ and is parallel to the vector $\vec{v} = \langle 1,-1,2\rangle$. A second line L_2 passes through the points $Q(0,2,2)$ and $R(1,1,2)$.

   a) Find the intersection point of the two lines.

   Solution: $L_1 = \langle 1+t, 1-t, 2t\rangle$ and the vector parallel to L_2 is $\vec{w} = \langle 1,-1,0\rangle$ so $L_2 = \langle s, 2-s, 2\rangle$. Equating the lines component wise we see $t=1$ and $s=2$ so the point of intersection is $(2,0,2)$.

   b) Find and equation of the plane $\mathcal{P}$ that contains these two lines.

   Solution: Let $\vec{n} = \vec{v}\times\vec{w} = \langle 2,2,0\rangle$. Then using the point P, the equation of the plane is $2(x-1)+2(y-1)+0(z-0)=0 \implies x+y=2$.

   c) Find the distance from the origin to the plane $\mathcal{P}$.

   Solution: The distance is given by

$$\frac{|0+0+0-2|}{\sqrt{1+1+0}} = \sqrt{2}.$$

10. Consider the surface $x = y^2+4y+z^2+5$. Sketch the surface and indicate your reasoning. Describe the cross-sections (traces) of the surface with planes perpendicular to any of the three axes, and make sure that these features are reflected in your sketch. Are there any special traces? What is the name of this surface? Include all relevant calculations and explanations.

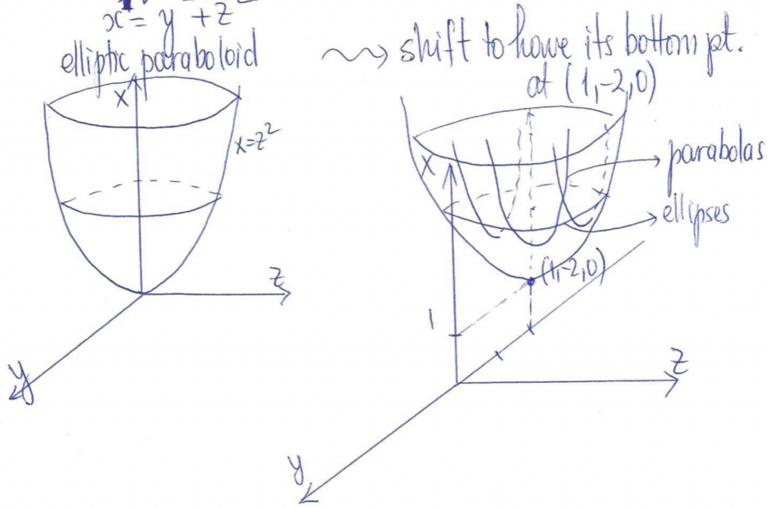
Solution:

$x = (y^2+4y+4)+z^2+1$

$x-1 = (y+2)^2+z^2$

$x = y^2+z^2$

elliptic paraboloid



Name of surface: elliptic paraboloid

Why? (x=k) hor. traces: ellipses; two vert. traces (y=k; z=k) parabolas

Special trace(s): X=1: pt. (1,-2,0)

11. Consider the surface $x^2-4y^2+z^2+4x-8y-6z=-13$. Sketch the surface and indicate your reasoning. Describe the cross-sections (traces) of the surface with planes perpendicular

to any of the three axes, and make sure that these features are reflected in your sketch. Are there any special traces? What is the name of this surface? Include all relevant calculations and explanations.

Solution:

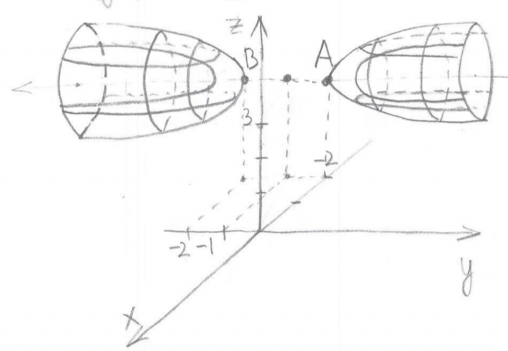
$$(x^2+4x+4)-(4y^2+8y+4)+(z^2-6z+9)=9-13$$

$$\iff (x+2)^2-4(y+1)^2+(z-3)^2=-4$$

$$\iff \left(\frac{x+2}{2}\right)^2-(y+1)^2+\left(\frac{z-3}{2}\right)^2=-1$$

$$\iff (y+1)^2-\left(\frac{x+2}{2}\right)^2-\left(\frac{z-3}{2}\right)^2=1 \rightsquigarrow u^2-v^2-w^2=1$$

hyperboloid of 2sheets



- $|y+1|=1 \Rightarrow x=-2$, $z=3$

$\Rightarrow$ for $y+1=\pm1$, $y=0,-2$ we get traces that are points:

$A=(-2,0,3)$ &

$B=(-2,-2,3)$

Name of traces ⊥ x-axis: hyperbola

Name of traces ⊥ y-axis: ellipse (circle)

Name of traces ⊥ z-axis: hyperbola

Traces that are points? Where? $(-2,0,3)$ & $(-2,-2,3)$

Name of surface: hyperboloid of 2 sheets

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4.3 Calculus with Vector-Valued Functions

We looked at vector functions, limits of vector functions, derivatives of vector functions, unit tangent vectors, differentiation laws (sum/difference rule, chain rule, product rules) for vector functions, integrals of vector-valued functions, arc length of vector-valued functions.

1. True/false practice:
   a) The domain of the function $\vec{u}(t)=\vec{v}(t)\cdot\vec{w}(t)$ is $[0,2]$ if the domain of $\vec{v}(t)$ is $t\geq 0$ and domain of $\vec{w}(t)$ is $t\leq 2$.

   Solution: True. The domain of $u(t)$ will be the intersection of the domains of $\vec{v}(t)$ and $\vec{w}(t)$, since for $\vec{v}(t)\bullet\vec{w}(t)$ to be defined we need both $\vec{v}(t)$ and $\vec{w}(t)$ to be defined, so the domain of u will be the intersection of the domains of $\vec{v}$ and $\vec{w}$.

   b) The sum of two differentiable vector functions is differentiable.

Solution: True. Since we differentiate vector functions by differentiating each of their components, we can apply the differentiation laws for scalar functions to each of the components to get differentiation laws for vector functions. In particular, this technique allows us to show that $\big(\vec{v}(t)+\vec{w}(t)\big)'=\vec{v}'(t)+\vec{w}'(t)$, since the sum of two differentiable scalar-valued functions is differentiable with derivative the sum of the derivatives of the two summands.

c) To find the definite integral from $t=a$ to $t=b$ of a vector function $\vec{r}(t)=\langle f(t),g(t),h(t)\rangle$, we simply take the vector whose components are the definite integral $\int_a^b f(t)dt$, $\int_a^b g(t)dt$, and $\int_a^b h(t)dt$, respectively.

Solution: True. Integration of vector functions is done componentwise.

d) The arc length between two points in $\mathbb{R}^3$ of a curve given by a vector function depends on the parameterization of the curve; if we replace t with $2t$, we'll double the arc length since the new curve is going twice as fast.

Solution: False. The arc length of a curve is *intrinsic*; i.e. it is independent of the parametrization. In particular, we can use integration by substitution to show that reparametrization doesn't change the value of the integral defining arc length.

e) When we renormalize a tangent vector to find a unit tangent vector to a curve at a given point, it is okay if we multiply by -1 since multiplying a unit vector -1 gives a unit vector so multiplying by -1 doesn't change anything.

Solution: False. The unit tangent vector captures the instantaneous direction of travel along a curve, so multiplying it by -1 would represent moving along the curve in the opposite direction.

2. Two flies move through space with flight paths described by the vector functions $\vec{r}_1(t)=\langle 2t+1,3+\cos\pi t,4t\rangle$ and $\vec{r}_2(t)=\langle t,3-\sin\pi t,t^2-2\rangle$.

a) At the time $t=3$, which fly is flying faster?

Solution: The speed of a particle moving along the path $\mathbf{r}(t)$ is $|\mathbf{r}'(t)|$. The speeds at $t=3$ are $\mathbf{r}_1'(3)$ and $\mathbf{r}_2'(3)$. We have

$$\mathbf{r}_1'(t)=\langle 2,-\pi\sin\pi t,4\rangle,\quad \mathbf{r}_2'(t)=\langle 1,-\cos\pi t,2t\rangle.$$

At time $t=3$, we have

$$\mathbf{r}_1'(3)=\langle 2,0,4\rangle,\quad \mathbf{r}_2'(3)=\langle 1,1,6\rangle.$$

These two vectors have lengths $\sqrt{2^2+0^2+4^2}=2\sqrt{5}$ and $\sqrt{1^2+1^2+6^2}=\sqrt{38}$. Since $\sqrt{38}>2\sqrt{5}$, we see that the second fly is moving faster than the first.

b) Do their paths intersect?

Solution: We look for a point $(x, y, z) = \mathbf{r}_1(t) = \mathbf{r}_2(s)$ for some (possibly distinct) times t and s, since we are only looking for a point on both curves. This gives us a system of equations

$$\begin{aligned} 2t + 1 &= s \\ 3 + \cos \pi t &= 3 - \sin \pi s \\ 4t &= s^2 - 2. \end{aligned}$$

Solving the first and third equations together, we see that $4t = (2t+1)^2 - 2 = 4t^2 + 4t - 1$, so that $4t^2 - 1 = 0$. This tells us that either $t = \frac{1}{2}$ (so that $s = 2$) or $t = -\frac{1}{2}$ (so that $s = 0$). Plugging these into the second equation, $3 + \cos \frac{\pi}{2} = 3$, $3 - \sin 2\pi = 3$, so the point $(2, 3, 2)$ is on both curves $\mathbf{r}_1(t)$ (at time $\frac{1}{2}$) and $\mathbf{r}_2(t)$ (at time $s = 2$). This tells us that the flies' paths $\boxed{\text{do intersect}}$.

c) Do the flies collide?

Solution: The flies don't necessarily collide just because their paths intersect; to have a collision, the flies need to be at the same point at the same time. In particular, if the flies are at the same x-coordinate at the same time, then we must have $2t + 1 = t$, so that $t = -1$, but the y-coordinates at that time are $3 - 1 = 2$ and $3 - 0 = 3$, so the two flies can't be at the same x and y coordinates at the same time. Thus they cannot be at the exact same point at any time; the flies $\boxed{\text{don't collide}}$.

3. Find symmetric equations for the tangent line to $\vec{r}(t) = \langle 3t^2, 1+e^{t-1}, t^{-1} \rangle$ at the point $(3, 2, 1)$.

Solution: The point $(3, 2, 1)$ occurs at $t = 1$, since $\frac{1}{t} = 1$ only when $t = 1$, and at $t = 1$ $3t^2 = 3$ and $1 + e^{t-1} = 2$. To find the tangent line, we need to find $\mathbf{r}'(1)$, as that vector will be parallel to the tangent line. We simply differentiate each of the three components to get that

$$\mathbf{r}'(t) = \langle 6t, e^{t-1}, -\frac{1}{t^2} \rangle.$$

In particular, this tells us that $\mathbf{r}'(1) = \langle 6, 1, -1 \rangle$. The tangent line to $\mathbf{r}(t) = \langle 3t^2, 1 + e^{t-1}, \frac{1}{t} \rangle$ ata the point $(3, 2, 1)$ is thus the line through $(3, 2, 1)$ parallel to $\langle 6, 1, -1 \rangle$. The parametric equations for such a line are $x = 3 + 6t, y = 2 + t, z = 1 - t$, so solving for t from each of the three equations, we get the symmetric equations

$$\boxed{\frac{x-3}{6} = y - 2 = -(z-1)}.$$

4. Consider three vector-valued functions $\vec{a}(t), \vec{b}(t), \vec{c}(t)$ taking values in $\mathbb{R}^3$. What is the derivative of the triple product $\vec{a}(t) \circ (\vec{b}(t) \times \vec{c}(t))$?

Solution: We use our product rules for vector products:

$$(\vec{u}(t) \bullet \vec{v}(t))' = \vec{u}'(t) \bullet \vec{v}(t) + \vec{u}(t) \bullet \vec{v}'(t)$$

and

$$(\vec{u}(t) \times \vec{v}(t))' = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t).$$

Combining these, we see

$$\begin{aligned}\left(\vec{a}(t) \bullet (\vec{b}(t) \times \vec{c}(t))\right)' &= \vec{a}'(t) \bullet (\vec{b}(t) \times \vec{c}(t)) + \vec{a}(t) \bullet (\vec{b}(t) \times \vec{c}(t))' \\ &= \vec{a}'(t) \bullet (\vec{b}(t) \times \vec{c}(t)) + \vec{a}(t) \bullet (\vec{b}'(t) \times \vec{c}(t) + \vec{b}(t) \times \vec{c}'(t)) \\ &= \boxed{\vec{a}'(t) \bullet (\vec{b}(t) \times \vec{c}(t)) + \vec{a}(t) \bullet (\vec{b}'(t) \times \vec{c}(t)) + \vec{a}(t) \bullet (\vec{b}(t) \times \vec{c}'(t))}.\end{aligned}$$

5. A particle moves with position vector described by $\vec{r}(t) = \langle \cos t, \sin t, \ln \sec t \rangle$ for $0 \leq t \leq \pi/4$. How far does the particle travel between $t = 0$ and $t = \pi/4$.

Solution: We have the arc length formula

$$\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

These componentwise derivatives are

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t, \quad \frac{dz}{dt} = \frac{sect \tan t}{\sec t} = \tan t,$$

so our arc length is given by

$$\begin{aligned}\int_0^{\pi/4} \sqrt{\sin^2 t + \cos^2 t + \tan^2 t} dt &= \int_0^{\pi/4} \sqrt{1 + \tan^2 t} dt \\ &= \int_0^{\pi/4} \sec t dt \\ &= \ln(\sec t + \tan t)\Big|_0^{\pi/4} \\ &= \ln(\sqrt{2} + 1) - \ln(1 + 0) \\ &= \boxed{\ln(\sqrt{2} + 1)}.\end{aligned}$$

6. A spaceship is traveling along the twisted cubic C with position function $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ for all $t \in \mathbb{R}$. Below, include all relevant calculations and explanations.

   a) Find the tangential component of the spaceship's acceleration along C at the origin.

   Solution: First notice that $r'(t) = \langle 1, 2t, 3t^2 \rangle$ and $r''(t) = \langle 0, 2, 6t \rangle$. These vectors evaluated at $t = 0$ are $\langle 1, 0, 0 \rangle$ and $\langle 0, 2, 0 \rangle$, respectively. Then

   $$a_T = \frac{r'(0) \cdot r''(0)}{|r'(0)|} = 0.$$

   b) Find the normal component of the spaceship's acceleration along C at the origin.

   Solution: $a_N = \frac{|r'(0) \times r''(0)|}{|r'(0)|} = \frac{|\langle 0,0,2 \rangle|}{1} = 2.$

c) Write the spaceship's acceleration vector at the origin in terms of the local coordinate unit vectors $\vec{T}$, $\vec{N}$, and $\vec{B}$.

Solution: $a = a_T\vec{T} + a_N\vec{N}$ so $a(0) = 0\vec{T} + 2\vec{N} = 2\vec{N}$.

7. Prove, using the methods we learned in this course, that the tangent line at a point A to a circle with center O is perpendicular to the radius OA.

Solution: Without loss of generality, we can assume our circle is centered at the origin by picking our coordinates to have the origin as the center of the circle. We write a vector-valued function in $\mathbb{R}^2$ representing the circle: $\mathbf{r}(t) = \langle r\cos t, r\sin t\rangle$ where r is the radius of the circle. We can find the tangent vector, $\mathbf{r}'(t)$, by differentiation: $\mathbf{r}'(t) = \langle -r\sin t, r\cos t\rangle$. The tangent line to the circle at a point is parallel to the tangent vector, so to show that the tangent line at a point A is perpendicular to the radius OA, it suffices to show that $\mathbf{r}'(t) \perp \mathbf{r}(t)$, since the segment OA is represented by the vector $\mathbf{r}(t)$ from the origin to the point. We evaluate the dot product $\mathbf{r}'(t) \bullet \mathbf{r}(t)$. We have

$$\begin{aligned}\mathbf{r}'(t) \bullet \mathbf{r}(t) &= \langle -r\sin t, r\cos t\rangle \bullet \langle r\cos t, r\sin t\rangle \\ &= -r^2\sin t\cos t + r^2\cos t\sin t \\ &= 0.\end{aligned}$$

So at all times t, and thus at all points on the circle, $\mathbf{r}'(t)$ is perpendicular to $\mathbf{r}(t)$. This shows that the radius OA is perpendicular to the tangent line at the point A.

4.4 Extra True/False Practice

1. True/false practice:

a) The vector function $\vec{r}(t) = \langle t, t^2\rangle$ contains more information than the Cartesian function $y = x^2$.

Solution: True. This vector function also tells us something about a direction of travel along the curve (from $x = -\infty$ to $x = +\infty$) that is not captured by $y = x^2$. Vector functions also have a speed $\mathbf{r}'(t)$ associated with them (since they give parametrizations of curves in the plane or in space) that simple Cartesian expressions lack.

b) Any three points in $\mathbb{R}^3$ determine a unique plane.

Solution: False. Three points that lie on the same line do not determine a unique plane; for example, both the planes $z = 0$ and $y = 0$ contain the three points $(0,0,0)$, $(1,0,0)$, and $(2,0,0)$.

c) Any time $dx/dt = 0$ for a parametric curve, the curve has a vertical tangent.

Solution: False. If we have $\frac{dy}{dt} = 0$ at that time as well, then we might have a horizontal tangent or a tangent of any other slope as well.

d) If $\vec{u}(t)$ and $\vec{v}(t)$ are vector functions, then $\vec{u}(t) \times \vec{v}(t)$ is a vector function.

Solution: True. The cross product of two vectors is a vector, so the cross product of two vector functions will be a vector function.

e) Any line in $\mathbb{R}^3$ will intersect one of the three coordinate planes $x = 0, y = 0, z = 0$.

Solution: True. A line is given by a vector equation of the form $\mathbf{r}(t) = \mathbf{r}_0 + t\vec{v}$, where $\vec{v}$ is a nonzero vector. In particular, one of the x, y, and z components of $\vec{v}$ must be nonzero, so that t times that component is going to be equal to the opposite of that component of r_0, and hence either x, y, or z will be zero at some point on the line.

f) Let P_1, P_2, and P_3 be three distinct planes in $\mathbb{R}^3$. Then the intersection of all three planes, $P_1 \cap P_2 \cap P_3$ is either the empty set, a single point, or a line.

Solution: True. The intersection $P_1 \cap P_2$ is either a line or the empty set, since P_1 and P_2 aren't the same plane. If $P_1 \cap P_2$ is a line, it can intersect the plane P_3 either in a line (if the line happens to lie in P_3), a point (if the line pierces through the plane P_3), or the empty set. This tells us that $P_1 \cap P_2 \cap P_3$ is either the empty set, a single point, or a line.

g) We say that the graph of $xz = 1$ in $\mathbb{R}^3$ is a cylinder even though the traces for constant y are hyperbolas and not circles.

Solution: True. Note that a cylinder is, for us, includes surfaces in $\mathbb{R}^3$ of the form $f(x, y) = 0$, $f(y, z) = 0$, and $f(x, z) = 0$ (as well as other surfaces where all the slices along a particular direction are identical curves), not just circular cylinders.

h) The arc length formula for a vector function $\vec{r}(t)$ in $\mathbb{R}^3$, $L = \int_\alpha^\beta |\vec{r}(t)|dt$, is true for the same reasons as our arc length formulas for cartesian, polar, and parametric curves even though, when written this way, there is no square root sign.

Solution: True. The length of the vector $\mathbf{r}'(t)$ is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$, which is analogous to our formulas for cartesian, polar, and parametric curves.

i) We can parametrize a hyperbola much the same way as we parametrize an ellipse, replacing cos with cosh and sin with sinh.

Solution: True. The curve $x = \cosh t$, $y = \sinh$, where cosh is the hyperbolic cosine function and sinh is the hyperbolic sine function, parametrize the hpyerbola $x^2 - y^2 = 1$.

j) if the lines with vector equations $\vec{r}_0 + t\vec{v}$ and $\vec{r}_1 + s\vec{w}$ are skew, there is *nothing* we can say about $\vec{v} \times \vec{w}$.

Solution: False. Since the lines are skew and not parallel, we know that $\vec{v} \times \vec{w}$ is nonzero, so we can say *something* about $\vec{v} \times \vec{w}$.

k) The centroid of a triangle can be found by intersecting just *two* of the medians of the triangle.

Solution: True. The three medians interesect at one point.

l) If a parametric curve given by $x = f(t)$, $y = g(t)$ satisfies $g'(0) = 0$, then the curve has a horizontal tangent at $t = 0$.

Solution: False. Let $x = t^3, y = t^3$ then $g'(0) = 0$ but $x = y$ is not horizontal.

m) The curve with vector equation $\vec{r}(t) = 6\,t^5\,\vec{i} - t^5\vec{j} + 3\,t^5\vec{k}$ is a line.

Solution: True $x = -6y, z = -3y$ and t^5 goes over $\mathbb{R}$ once

n) Among quadric surfaces, there are *three* types of "elliptic hyperboloids," but none of them is called by this name.

Solution: True Cone and hyperboloid of one sheet and two sheets.

o) The polar curves $r = 1 - \cos(2\theta)$ and $r = \cos(2\theta) - 1$ have the same graph.

Solution: True

$$(r, \theta) = (-r, \theta + \pi) = (\cos 2\theta - 1, \theta + \pi) = (\cos(2(\theta + \pi)) - 1, \theta + \pi) = (\cos 2\theta_1 - 1, \theta_1)$$

p) For any vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ we have $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w}$.

Solution: False the cross product is not associative.

q) If $\vec{v}$ and $\vec{w}$ are vectors in the plane, then the equality $|\vec{v} + \vec{w}| = |\vec{v}| + |\vec{w}|$ is satisfied only when $\vec{v}$ and $\vec{w}$ are in a special position relative to each other.

Solution: True Equality iff $\vec{v}//\vec{w}$.

r) There are at least *three* "product rules" for differentiation that involve vector functions.

Solution: True Dot product, cross product, scalar function multiple of vector function.

s) If $|\vec{r}(t)| = 7$ for all t, then $\vec{r}\,'(t) \perp \vec{r}(t)$ for all t.

Solution: True proved in class

t) Different parameterizations of the same smooth curve result in identical *unit tangent* vectors on the curve.

Solution: False reverse direction results in negative of the unit tangent.

u) Given a curve $x = f(t)$ and $y = g(t)$, with both $f'(t)$ and $g'(t)$ continuous and both $f(t) \geq 0$ and $g(t) \geq 0$ for $a \leq t \leq b$, the two formulas for the *surface areas* of the solids of revolution about the x-axis and the y-axis differ only in one place.

Solution: True One integrand has $2\pi x$ and the other has $2\pi y$.

v) It is straightforward to convert from *Cartesian* to *polar* coordinates; but we have to pay attention to the quadrant where the point is when converting from *polar* to *Cartesian* coordinates because we could end up with the wrong angle θ.

Solution: False The terms are switched

w) A shortcut formula for the *tangential* component of the acceleration vector can be turned into a shortcut formula for the *normal* component by changing one vector multiplication operation to another in the numerator and adding absolute values there to ensure that we will obtain overall a number.

Solution: True Changing the dot product to cross product.

x) The equation defining any *cylinder* in $\mathbb{R}^3$ is necessarily missing one of the variables x, y, or z that corresponds to the ruling of the cylinder.

Solution: False If the cylinder does not have center axis parallel to a coordinate axis then we could have all three variables.

y) *Exactly one* of the following two statements is true:
(a) $|\vec{v} \circ \vec{w}| = |\vec{v}| \cdot |\vec{w}|$ for *some* vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$;
(b) $|\vec{v} - \vec{w}| = |\vec{v}| - |\vec{w}|$ for *all* vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$.

Solution: True. The first statement is true if the vectors are parallel or one of the is the zero vector. The second statement is only true for antiparallel vectors so not *all* vectors.

5 No Calculators during the Exam. Cheat Sheet and Studying for the Exam

No Calculators will be allowed during the exam. Anyone caught using a calculator will be disqualified from the exam.

For the exam, you are allowed to have a "cheat sheet" - *one page* of a regular 8.5×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.) DSP students with special writing or related disability should consult with the instructor regarding their cheat sheets.
- You must submit your cheat sheet on Gradescope **before** the exam.
- Any violation of these rules will disqualify your cheat sheet and may end in your own disqualification from the exam. I may decide to randomly check your cheat sheets, so let's play it fair and square. :)

- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.
- **Do NOT overstudy on the day of the exam!! No sleeping the night before the exam due to cramming, or more than 3 hours of math study on the day of the exam is counterproductive! No kidding!**